greatly superior to those of ordinary construction; but the number of them being comparatively limited, it would perhaps be sufficient to treat them as one class of risk until information were collected from which certain statistics might be derived.

The application of the average clause in practice presents so many anomalies and contradictions, that the data derived from insurances under its operation differ widely from those which would be obtained were the principle of average carried out in its integrity. Many insurances are, however, effected subject to the average clause, especially upon goods in mercantile warehouses; and these might with propriety be arranged in classes, as the data respecting them must yield pretty conclusive results.

Hitherto the Offices have been contented with a very rough estimate of the risks incurred and the premiums which should be charged; and it is doubtful whether, since fire insurance was established as a business, any real advancement has been made in the methods of procuring statistics. Those which are derived from sources foreign to the Office records have little value for an Insurance Company, as they cannot give the means for *comparing* the number of risks with the value of property and the actual damage which occurs. But as the basis upon which the business is founded is statistics, it is important that these should be correct; and there is no reason why the Offices should rest contented with imperfect data, when their own books can supply them accurately, and in the greatest abundance; and if some experienced members of the profession could be induced to co-operate in arranging a comprehensive and simple classification of risks, the individual Offices or such of them as approved of the idea might analyze their own business in conformity with that model; and were it thought advisable, their united statistics might be collected for the benefit of the profession generally.

Continuation of a Paper on a Method of Distributing the Surplus among the Assured in a Life Assurance Company. *By* T. B. SPRAGUE, M.A., *Fellow of St. John's College, Cambridge.*

IN the present paper I propose to give some numerical illustrations of the working of the method of which the formulæ were given in the last Number of this *Magazine*, and to add some further results which have been subsequently arrived at.

In order to apply the formulæ to numerical cases, it is necessary to assume some law according to which the successive bonuses proceed. It is obvious, that whatever law is assumed must be to a great extent arbitrary, as it is of course quite impossible to say what will be the state of a particular Office after the lapse of a number of years. But it seems probable that, on an average, an Office may return to the assured in the form of bonus 20 per cent. of the premiums paid, and it is accordingly assumed in the calculations that each quinquennial cash bonus declared on a policy is equal in amount to one year's premium. This will probably not be far from the truth where the method of division of profits adopted in the Office is one that leads to nearly constant cash bonuses to each policy: such as the method proposed by Mr. Jellicoe, which divides the profits in proportion to the loading of the premium, accumulated at compound interest for the interval between two successive bonuses. But in cases where one of the more common methods of division is adopted, which lead to a series of increasing cash bonuses on each policy, the above illustration will not be applicable, and the conditions of the problem will be greatly changed. The calculations are made by means of the "Experience" mortality, assuming interest at 4 per cent.

For the sake of more complete comparison, the various cases have been considered in which the assurance is commenced at different ages, from 20 up to 50, at intervals of five years; and in each case it is supposed that the first bonus is declared after a lapse of five years. The following table (page 346) exhibits, in column a, the effect of applying the former of the two methods explained in the paper already referred to.

From this table it will be seen, that on the suppositions already specified, if a person commence an assurance at the age of 20, the effect of the first bonus, declared when he has reached the age of 25, will be to relieve him from all further payments after attaining the age of 64. The effect of the second bonus, when his age will be 30, is to relieve him from all payments after attaining the age of 59; and so the age at which his payments cease is reduced by successive bonuses to 55 and 53, the final result being that his payments cease after the age of 52. He will thus have paid thirty-three premiums; and, taking the Northampton 3 per cent. rates, and premium for age 20 at £2. 17*s.* 9*d.*, we see that it will require forty premiums to amount to the sum assured.

Age at commencing Assurance	20		25		30		35		40		45		50	
	α.	*β.*	*α.*	*β.*	*α.*	*β.*	*α.*	*β.*	*α.*	*β.*	*α.*	*β.*	*α.*	*β.*
1st Bonus.....	64	..	66	..	68	..	70	..	71	..	73	..	75	..
2nd „	59	..	61	..	63	..	65	..	67	..	69	..	71	..
3rd „	55	..	58	..	60	..	63	..	65	..	67	..	70	..
4th „	53	..	56	..	59	..	61	..	64	..	66	..	..	79
5th „	52	..	55	..	57	..	..	77	..	79	..	80	..	77
6th „	..	77	..	78	..	79	..	75	..	76	..	78		
7th „	..	74	..	75	..	77	..	74	..	75				
8th „	..	72	..	74	..	76								
9th „	..	71	..	73										
10th „	..	70												

NOTE.—If interest be taken at 3 per cent., the ages under 20 in column *α* will be 68, 63, 59, 57, 55, 54; and in column *β*, 78, 75, 73, 72.

Two remarks require to be made upon the method followed in calculating the ages given in the above table. In the first place, the value of N_{m+x} given by the formula

$$N_{m+x}=\frac{C}{P}.D_m \quad . \quad . \quad . \quad (1)$$

will not in general correspond exactly with any value of N in the table, but will lie between consecutive values, as N_p and N_{p+1}: the rule that has been adopted is, to take always the higher age $p+1$. Again: in calculating the effect of the second bonus by means of the formula

$$N_y=N_{m+x}+\frac{C'}{P}.D_{m+5} \quad . \quad . \quad (2),$$

it seems more equitable to the assured to take the value of N_{m+x}, as given by the formula (1), than the smaller value N_{p+1}, and this has accordingly been done throughout; or, virtually, the formula used has been $N_y=\frac{CD_m+C'D_{m+5}}{P}$.

The ages given in column β of the above table will be explained below.

If this method of distributing profits were adopted to any extent in an Office, there is no doubt the labour of the periodical valuations would be increased; but with the help of complete tables of deferred annuities, such as those given by Mr. Thomson, it does not appear that the labour would be much greater than that involved in the method frequently adopted of valuing each policy separately. Thus, a policy on which payment of premium is to

cease after the age p, might be valued as an ordinary one, and be included in the appropriate list of policies, if the method of valuation in classes be adopted, a deduction being subsequently made for the value of the premiums after the age p.

There seems one case in which this method may be very appropriately applied: I refer to the practice of calculating a bonus for a policy, but not allowing that bonus to vest till the policy has been a certain number of years in force. If the portion of profits allowed to a policy were applied in this way, the condition is, from the nature of the case, satisfied.

In discussing the second part of the problem proposed in the last Number of the *Magazine*, I arrived at the equation

$$S(D_{m+x}-M_{m+x})+P.N_{m+x-1}=C.D_m \quad . \quad (3)$$

This equation I was not at the time in a position to solve; and I am indebted to Mr. Samuel Younger, of the Engineers' Life Office, for the following elegant solution. We have

$$D_m+N_m=N_{m-1}$$
$$M_m+N_m=rN_{m-1},$$

whence

$$D_m-M_m=(1-r)N_{m-1}=dN_{m-1} \quad . \quad . \quad (4)$$

r denoting as usual the present value of £1 due in one year, and d the discount on £1 for a year. Substituting in the equation (3), we get

$$SdN_{m+x-1}+PN_{m+x-1}=C.D_m,$$

and

$$N_{m+x-1}=\frac{C.D_m}{Sd+P} \quad . \quad . \quad . \quad (5)$$

whence x is determined as in former cases.

We can now proceed to determine the effect of a second bonus upon the age at which the sum assured becomes payable. Thus, making the same suppositions as previously, that a second cash bonus of C′ is declared after an interval of five years—suppose that the effect of this is to make the sum assured payable at the age y instead of $m+x$. The age of the life assured is now $m+5$; and the value of £1, payable at age $m+x$ or previous death, is

$$\frac{M_{m+5}+D_{m+x}-M_{m+x}}{D_{m+5}},$$

and the value of £1 payable at the age y or previous death is

$$\frac{M_{m+5}+D_y-M_y}{D_{m+5}}:$$

the difference between the two is

$$\frac{D_y - M_y - D_{m+x} + M_{m+x}}{D_{m+5}},$$

which, by the application of formula (4), is equal to

$$\frac{dN_{y-1} - dN_{m+x-1}}{D_{m+5}}.$$

Again, the value of the deferred temporary annuity from age $y-1$ to $m+x-1$ is $\frac{N_{y-1} - N_{m+x-1}}{D_{m+5}}$. Hence the whole benefit to the assured in consequence of having the sum assured made payable at the age y instead of $m+x$ will be

$$S\frac{dN_{y-1} - dN_{m+x-1}}{D_{m+5}} + P\frac{N_{y-1} - N_{m+x-1}}{D_{m+5}},$$

which is to be equal to C′.

This equation gives us

$$\frac{Sd+P}{D_{m+5}}(N_{y-1} - N_{m+x-1}) = C',$$

and

$$N_{y-1} = N_{m+x-1} + \frac{C'D_{m+5}}{Sd+P} \quad . \quad . \quad (6)$$

whence y is determined.

The following table exhibits the effect of this method of distributing the profits upon the age at which the assurance is made payable. The same suppositions have been made as in the former case, as to the magnitude of the successive cash bonuses, and the same rules observed in calculating the ages given in the table. The rates charged are supposed to be the Northampton. The calculations have also again been made on the suppositions of the Experience mortality and a rate of interest of 4 per cent.

Age at which the Assurance is commenced	20	25	30	35	40	45	50
1st Bonus.....	73	74	75	76	77	78	79
2nd „	69	70	71	72	74	75	76
3rd „	66	68	69	70	72	73	75
4th „	65	66	68	69	71	72	74
5th „	64	65	67	68	70		
6th „	63	65	66				
7th „	62	64					

A comparison of the results given by the application of the two methods, shows that the benefit derived by the assured from the

application of the second method is much more remote than that derived from the application of the first.

In order to compare the results of the above methods more completely with those commonly in use, the equivalent reductions of premium at successive divisions of profits have been calculated on the suppositions already specified, and the ages ascertained at which the payment of premium will entirely cease; the results being set forth in the subjoined table.

Age at which the Assurance is made.	Age at which the Premiums cease.
20	75
25	75
30	80
35	80
40	80
45	80
50	85

Now, suppose that, in consequence of the application of the former method, the life assured has reached the age at which the payment of premium ceases, and that the age at the next division of profits is p; then, according to the custom of some Offices, a bonus will still be allowed to the assured, of the same magnitude as if the payment of premium had continued: we may apply this bonus, in accordance with the principles already adopted, to make the sum assured payable during the life of the assured. Thus, suppose that a cash bonus C is allotted to the policy, and that its effect, when applied in the way mentioned above, is to make the sum assured payable at the age x; then, pursuing the same method as in former cases, the formula will be $\mathrm{S}.\frac{\mathrm{D}_x-\mathrm{M}_x}{\mathrm{D}p}=\mathrm{C}$, or $\mathrm{S}.\frac{d\,\mathrm{N}_{x-1}}{\mathrm{D}p}=\mathrm{C}$, and

$$\mathrm{N}_{x-1}=\frac{\mathrm{C}.\mathrm{D}p}{\mathrm{S}d} \quad . \quad . \quad . \quad (7)$$

If a second cash bonus of C′, after an interval of five years, be applied in the same way, so that the sum assured will be payable at the age y or previous death; the formula is

$$\mathrm{S}.\frac{\mathrm{D}_y-\mathrm{M}_y-\mathrm{D}_x+\mathrm{M}_x}{\mathrm{D}_{p+5}}=\mathrm{C}',$$

or

$$\mathrm{S}d\frac{\mathrm{N}_{y-1}-\mathrm{N}_{x-1}}{\mathrm{D}_{p+5}}=\mathrm{C}',$$

and

$$N_{y-1}=N_{x-1}+\frac{C'.D_{p+5}}{Sd} \quad . \quad . \quad (8)$$

It is obvious that the formulæ (7) and (8) may be deduced from (5) and (6), by putting $P=O$, and writing p for m; and a little consideration shows that this should be the case. The figures in column β of the table at page 346 exhibit the application of this method to the cases to which the first method has been previously applied.

Or again, suppose that, by the application of the second method, the assurance has been made payable at the age p; and the age of the life, at the next division of profits, is m; and, let it be required to find the age (y) at which the payment of premiums will be made to cease in consequence of a cash bonus C: the formula for this case will be $P.\frac{N_y-N_{p-1}}{D_m}=C$, and $N_y=N_{p-1}+\frac{C.D_m}{P}$.

The formula (5) may be established by general reasoning in the following manner, which may not be thought without interest.

Suppose that an endowment assurance is payable at the age p, and that the life has completed $p-1$ years, then the advantage to the assured, in consequence of the assurance being made payable at the age $p-1$ instead of p, is $Sd+P$; since the assured will be relieved from payment of the premium just due, and the sum assured will be paid a year earlier than it otherwise would. If now the life has completed m years, the present value of the above advantage is $\frac{D_{p-1}}{D_m}(Sd+P)$: or, this is the cash payment in consideration of which the terms of the assurance may be altered, so that the assurance shall be payable at the age $p-1$ instead of p. If, again, the original assurance be for the whole term of life, the benefit to the assured of having it converted into an assurance payable at age p or previous death is a deferred annuity of $Sd+P$, the first payment being made at age p. The value of this annuity is $\frac{N_{p-1}}{D_m}(Sd+P)$, m being the present age of the life; and, equating this to C, we get $N_{p-1}=\frac{C.D_m}{Sd+P}$.

The formula (6) may be established in a similar manner, substituting for the deferred annuity above a temporary deferred annuity.